

# Robust Decentralized Control - A Linear Matrix Inequality(LMI)-Based Design

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**Abstract**— This paper investigates decentralized control for linear time-invariant systems with norm-bounded time-varying structured uncertainty. We derive LMI conditions ensuring robust stability. We also show how to incorporate performance robustness, where performance is measured by the  $H_\infty$  gain of the decentralized system. The potential of the proposed technique has been demonstrated by a simulation example of three coupled inverted pendulums.

**Index Terms**—Decentralized control, LMI, norm-bounded uncertainty, robust stability.

## I. INTRODUCTION

A large scale system is often decomposed into a set of inter-connected subsystems to enable control implementation. Thus, research on decentralized control has been attracted the attention of many researchers since the seventies. A sufficient condition [1] for connective stability in terms of bounds of uncertain inter-connections between subsystems was studied. This connective stability laid the foundation of decentralized control. In [2] and [9]-[10] considered LTI systems with time-varying but bounded parametric uncertainties with the assumption that the uncertainties are in the range space of input matrix (matching condition). In [3] and [8], the matching conditions of the system are not valid. Thus the authors decomposed the uncertainties into matched and mismatched parts. The matched portion can be dealt with by feedback control while the mismatched portion was constrained by norm conditions in terms of state signals. In [4], the system with time-varying, norm-bounded parametric uncertainties with no matching assumption were studied. Sufficient conditions for stability were established in terms of a scaled  $H_\infty$  control system and a robust  $H_\infty$  decentralized controller was designed accordingly.

Note that decentralized control design is, in general, more difficult than that of centralized control. The difficult arises due to the fact that the controller for each subsystem is restricted to only using the local state or measurements in order to stabilize (with prescribed performance) the overall system. As in most control systems, a mathematical model usually cannot describe a dynamic system exactly. Also, a dynamical system is mostly working in a changing environment. This is especially true for large scale systems.

The uncertainties considered in this paper consist of real time-varying norm-bounded structured uncertainties that are realized by a linear fraction representation (LFR) in state space framework. LFR has gained significant attention due to its structured representation in terms of matrices [5]. The LFR

consists of an LTI system, connected with a diagonal feedback element depending on the setting of uncertainties. No “matching condition” is imposed on the uncertainties because of the system representation. We present LMI-based system analysis and controller synthesis in which Lyapunov matrix inequalities in terms of constraints can be cast as a convex optimization problem [7], [8].

The paper’s outline is as follows. In Section 2, we introduce the notation used in the paper and describe the system to be analyzed and controlled. Also, we define the decentralized control problem considered in the paper, including the stability and performance robustness. The performance is measured by  $L_2$  gain. Section 3 is devoted to the analysis of the open loop system, based on the robustness requirement. The tools for analysis are Lyapunov stability theory and convex optimization techniques. In Section 4, we address the corresponding controller synthesis problems for static state-feedback. In Section 5, we illustrate our methods on three inverted pendulums control problems.

## II. PROBLEM FORMULATION

### A. Notation

Throughout the paper the subscript “ $i$ ” denotes  $i$ th subsystem.  $I$  denotes the identity matrix with its size determined from context.  $q^r$  denotes the  $r$ th element of the vector  $q$  and  $\text{diag}(A, B)$  is defined as

$$\text{diag}(A, B) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

We reserve the subscript “ $d$ ” to express

$$A_d := \text{diag}(A_i)_{i=1,\dots,n} \quad A_i \in R^{n_i \times n_i}$$

The subscript “ $D$ ” is for

$$B_D := \text{diag}(B_i)_{i=1,\dots,n} \quad B_i \in R^{n_i \times m_i}$$

### B. Problem to Solve

The purpose of this subsection is to define a framework on which our approach to decentralized control is based. Consider a class of linear time-invariant continuous-time systems  $\Sigma$  with norm-bounded time-varying structured uncertainty, composed of  $n$  subsystems  $\Sigma_i$ .

$$\Sigma := \bigcup_{i=1}^n \Sigma_i(\underline{x}, u_i, p_i, w_i)$$

where  $\underline{x}$  is the state of system  $\Sigma$  and is defined as

$$\underline{x} = \begin{bmatrix} x_1^T & \cdots & x_i^T & \cdots & x_n^T \end{bmatrix}^T, \quad x_i \in R^{n_i}$$

The subsystem  $\Sigma_i$  is defined as follows.

$$\begin{aligned}\dot{x}_i &= A_i^i x_i + \sum_{j=1, j \neq i}^n A_i^j x_j + B_{u,i} u_i + B_{p,i} p_i + B_{w,i} w_i, \\ z_i &= C_{z,i}^i x_i + D_{zu,i} u_i + D_{zw,i} w_i, \\ q_i &= C_{q,i}^i x_i + \sum_{j=1, j \neq i}^n C_{q,i}^j x_j + D_{qu,i} u_i \\ &\quad + D_{qp,i} p_i + D_{qw,i} w_i, \\ p_i^r &= \delta_i^r(t) q_i^r, \quad |\delta_i^r(t)| \leq 1 \quad r = 1, 2, \dots, n_{q_i}\end{aligned}\quad (1)$$

where  $x_i$  denotes the state of subsystem  $i$ ,  $x_j$  is the state that acts on subsystem  $i$  from subsystem  $j$ ,  $u_i$  is the decentralized control input, signal  $w_i$  is the external inputs which include any unknown disturbance to be rejected as well as the reference command to be tracked.  $z_i$ , the controlled variable, may include tracking error or a cost of the input  $u_i$ . Assuming that the system is full state-feedback, that is  $y_i = x_i$ . The matrices  $A$ ,  $B$ ,  $C$ , and  $D$  with all superscripts and subscripts shown in (1) are assumed to be real constant matrices with appropriate dimensions. If  $x_j$  has no influence on  $x_i$ , the associated system matrix  $A_i^j$  will be set to zero. Notice that  $p_i$  and  $q_i$  represent the output and input vectors of system perturbations while the  $p_i^r$  and  $q_i^r$  correspond to the  $r$ th element of  $p_i$  and  $q_i$  respectively.

$$\begin{aligned}p_i &= \begin{bmatrix} p_i^1 & \cdots & p_i^r & \cdots & p_i^{n_p} \end{bmatrix}, \quad p_i^r \in R \\ q_i &= \begin{bmatrix} q_i^1 & \cdots & q_i^r & \cdots & q_i^{n_q} \end{bmatrix}, \quad q_i^r \in R\end{aligned}$$

$\delta_i^r(t)$  is the real valued unknown but bounded uncertainty and it is assumed to be a time-varying scalar quantity. Let

$$\Delta_{i,d} := \text{diag}(\delta_i^r(t))_{r=1, \dots, n_{q_i}}, \quad \delta_i^r(t) \in R.$$

It is assumed that the uncertainty  $\Delta_{i,d}$  belongs to a compact set

$$\mathcal{B}_\Delta := \left\{ \Delta_{i,d} : \|\Delta_{i,d}\| \leq 1, \forall i \right\}$$

where the prefix  $\mathcal{B}$  stands for “ball” or “bound”. Notice that the above-mentioned bounds applied in (1) can also be replaced by  $p_i^T p_i \leq q_i^T q_i$ . The state feedback closed-loop system of (1) is shown in Fig. 1 which mainly gives one the idea and meaning of how the system input, output, and uncertainty are represented. In Fig. 1 the labeled  $\Sigma_i$  represents the open-loop interconnection and contains all of the known elements including the nominal plant model, performance and the uncertainty weighting functions. The  $\Delta_{i,d}$  block contains all the uncertain elements and is assumed to be within a compact set  $\mathcal{B}_\Delta$  which parametrizes all of the parametric uncertainty in the problem. There are four sets of inputs entering into  $\Sigma_i$ : perturbation output  $p_i$ , disturbance  $w_i$ , time varying state inputs from other subsystem  $x_j$  and  $j \neq i$ , and decentralized controls  $u_i(t)$ . Three sets of outputs are: perturbation input  $q_i$ , controlled variable  $z_i$ , system state  $x_i$ , and state feedback gain  $K_i$ .

The robustness specifications we address in this paper are that the  $L_2$  gain of input/output signals for each individual subsystem is less than a specific number  $\gamma_i$ , which corresponds to the peak gain of the freq. response  $G_i(j\omega)$ , i.e.

$$\|G_i(j\omega)\|_\infty = \sup_\omega \bar{\sigma}(G_i(j\omega))$$

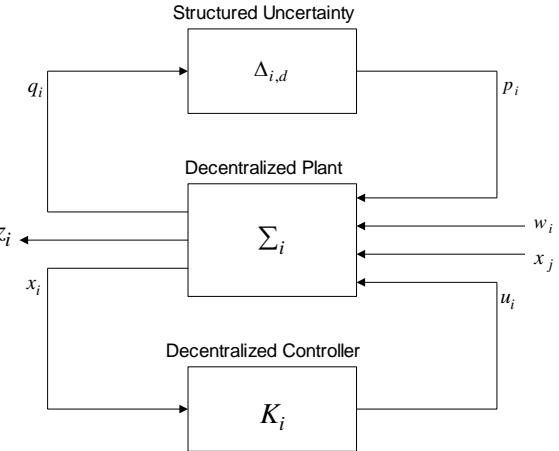


Fig. 1 Closed-loop for subsystem  $i$

where  $\|G_i\|_\infty$  represents the  $H_\infty$  norm of transfer function matrix from disturbance  $w_i$  to controlled variable  $z_i$ . Thus, according to the system described in (3.1) and the required robustness properties the following robust decentralized control problem is addressed.

Find a decentralized feedback controller  $u_i(t)$  as a linear function of local state vector  $x_i(t)$ , i.e.  $u_i(t) = K_i x_i(t)$  in which  $K_i$  is the feedback gain for subsystem  $i$  such that the resulting closed-loop system is robustly stable with respect to decentralized  $H_\infty$  disturbance attenuation for all admissible perturbations which lie in a compact set  $\mathcal{B}_\Delta$ . Sufficient conditions are

- (1) Closed-loop system is asymptotic stable.
- (2)  $\|G_i\|_\infty < \gamma_i$ .

### III. SYSTEM ANALYSIS

One of the powerful methods to analyze system stability is Lyapunov stability theory. Although there is no trivial method to establish a Lyapunov function, by experience, among Lyapunov functions that have been suggested and extensively used, the quadratic Lyapunov functions, i.e.  $V(\xi) = \xi^T L \xi$ , have been proved to be efficient and easily implemented [7].

Let's consider an equivalent system of (1)

$$\begin{aligned}\dot{x}_i &= A_i^i x_i + \sum_{j=1, j \neq i}^n A_i^j x_j + B_{p,i} p_i + B_{w,i} w_i, \\ z_i &= C_{z,i}^i x_i + D_{zw,i} w_i, \\ q_i &= C_{q,i}^i x_i + D_{qp,i} p_i + D_{qw,i} w_i, \\ p_i^T p_i &\leq q_i^T q_i.\end{aligned}\quad (2)$$

which is denoted by  $\Pi_i$ . Therefore, we have the Theme 1.

**Theme 1** The overall system  $\Pi := \bigcup_{i=1}^n \Pi_i(x_i)$  is robust decentralized stable with respect to  $H_\infty$  disturbance attenuation if there exists a symmetric matrix  $L_d$  such that the following matrix inequalities are satisfied  
 $L_d > 0$ ,  $\Lambda_d > 0$ ,  $\Gamma_d > 0$ , and

$$\begin{bmatrix} \text{BLK11} & \text{BLK12} & \text{BLK13} \\ (\text{BLK12})^T & \text{BLK22} & 0 \\ (\text{BLK13})^T & 0 & \text{BLK33} \end{bmatrix} < 0 \quad (3)$$

where

$$\text{BLK11} = \begin{pmatrix} A_d^T L_d + L_d A_d + L_d B_{a,D} B_{a,D}^T L_d \\ + C_{z,D}^T C_{z,D} + C_q^T \Lambda_d C_q + \Omega_d \end{pmatrix}$$

$$\text{BLK12} = L_d B_{p,D} + C_q^T \Lambda_d D_{qp,D}$$

$$\text{BLK22} = D_{qp,D}^T \Lambda_d D_{qp,D} - \Lambda_d$$

$$\text{BLK13} = L_d B_{w,D} + C_{z,D}^T D_{zw,D}$$

$$\text{BLK33} = D_{zw,D}^T D_{zw,D} - \Gamma_d.$$

*Proof.* See appendix A.

**Remarks:**

- 1) The last inequality in (3) is a quadratic inequality in matrix variable  $L_d$  can be further decomposed into LMI by Schur complement [7], [8] as shown in (4).

$$\begin{bmatrix} \text{BLK11} & \text{BLK12} & \text{BLK13} & L_d B_{a,D} \\ (\text{BLK12})^T & \text{BLK22} & 0 & 0 \\ (\text{BLK13})^T & 0 & \text{BLK33} & 0 \\ (L_d B_{a,D})^T & 0 & 0 & -I \end{bmatrix} < 0 \quad (4)$$

where

$$\text{BL11} = (A_d^T L_d + L_d A_d + C_{z,D}^T C_{z,D} + C_q^T \Lambda_d C_q + \Omega_d).$$

- 2) A dual condition of (4) can be established by a change of variable. Let  $Q_d = L_d^{-1}$ , the condition (4) can be equivalently written as

$$\begin{bmatrix} \text{QBLK11} & \text{QBLK12} & \text{QBLK13} & Q_d \\ (\text{QBLK12})^T & \text{QBLK22} & 0 & 0 \\ (\text{QBLK13})^T & 0 & \text{QBLK33} & 0 \\ Q_d^T & 0 & 0 & -H_d \end{bmatrix} < 0 \quad (5)$$

where

$$\text{QBLK11} = \begin{pmatrix} Q_d A_d^T + A_d Q_d + B_{a,D} B_{a,D}^T \\ + B_{p,D} M_d B_{p,D}^T + B_{w,D} N_d B_{w,D}^T \end{pmatrix},$$

$$\text{QBLK12} = B_{p,D} M_d D_{qp,D}^T + Q_d C_q^T,$$

$$\text{QBLK13} = B_{w,D} N_d D_{zw,D}^T + Q_d C_{z,D}^T,$$

$$\text{QBLK22} = -(M_d - D_{qp,D} M_d D_{qp,D}^T),$$

$$\text{QBLK33} = -(I - D_{zw,D} N_d D_{zw,D}^T),$$

$M_d = \Lambda_d^{-1}$ ,  $N_d = \Gamma_d^{-1}$ , and  $H_d = \Omega_d^{-1}$ . In fact, decentralized stability with performance requirement can be implied by the existence of  $Q_d > 0$  satisfying  $N_d > 0$ ,  $M_d > 0$ , and (5).

- 3) The well-posed condition for (2) is incorporated in (4) or (5), since it implies

$$\begin{bmatrix} \text{BLK22} & 0 \\ 0 & \text{BLK33} \end{bmatrix} < 0$$

or, equivalently,

$$\begin{bmatrix} \text{QBLK22} & 0 \\ 0 & \text{QBLK33} \end{bmatrix} < 0 \quad (6)$$

- 4) From Theme 1 and Remark 1) and 2), we see that a sufficient condition to determine system stability and performance can, in turn, be cast into convex optimization problem that is shown in [7], [8]. The problem can be easily formalized as an LMI feasible problem or can be cast into eigenvalue problem (EVP) as follows.

Maximize  $\text{Trace } N_d$   
Subject to  $Q_d = Q_d^T > 0$ ,  $N_d > 0$ ,  $M_d > 0$ , and (5) with well-posed condition (6).

#### IV. CONTROLLER SYNTHESIS

Consider the subsystem  $\Sigma_i$  with static state-feedback controller  $u_i(t) = K_i x_i(t)$ . The decentralized closed-loop system can be written as

$$\begin{aligned} \dot{x}_i &= (A_i^i + B_{u,i} K_i) x_i + \sum_{j=1, j \neq i}^n A_i^j x_j + B_{p,i} p_i + B_{w,i} w_i \\ z_i &= (C_{z,i}^i + D_{zu,i} K_i) x_i + D_{zw,i} w_i \\ q_i &= (C_{q,i}^i + D_{qu,i} K_i) x_i + \sum_{j=1, j \neq i}^n C_{q,j}^i x_j \\ &\quad + D_{qp,i} p_i + D_{qw,i} w_i \end{aligned} \quad (7)$$

Thus, the overall closed-loop system may be written as

$$\begin{aligned} \dot{x} &= (A_d + B_{u,D} K_D) \underline{x} + B_{a,D} \underline{x} + B_{p,D} p + B_{w,D} w \\ z &= (C_{z,D} + D_{zu,D} K_D) \underline{x} + D_{zw,D} w \\ q &= (C_q + D_{qu,D} K_D) \underline{x} + D_{qp,D} p + D_{qw,D} w \end{aligned} \quad (8)$$

In the view of Theme 1 and its remarks, the sufficient condition for system (7) to have robust decentralized  $H_\infty$  disturbance attenuation is

$Q_d > 0$ ,  $M_d > 0$ ,  $N_d > 0$ , and

$$\begin{bmatrix} \text{KBLK11} & \text{KBLK12} & \text{KBLK13} & Q_d \\ (\text{KBLK12})^T & \text{QBLK22} & 0 & 0 \\ (\text{KBLK13})^T & 0 & \text{QBLK33} & 0 \\ Q_d^T & 0 & 0 & -H_d \end{bmatrix} < 0 \quad (9)$$

where

$$\text{KBLK11} = \begin{pmatrix} Q_d (A_d + B_{u,D} K_D)^T + (A_d + B_{u,D} K_D) Q_d \\ + B_{a,D} B_{a,D}^T + B_{p,D} M_d B_{p,D}^T + B_{w,D} N_d B_{w,D}^T \end{pmatrix}$$

$$\text{KBLK12} = B_{p,D} M_d D_{qp,D}^T + Q_d (C_q + D_{qu,D} K_D)^T$$

$$\text{KBLK13} = B_{w,D} N_d D_{zw,D}^T + Q_d (C_{z,D} + D_{zu,D} K_D)^T.$$

Note that the last inequality in (9) is not convex since it jointly depends on  $Q_d$  and  $K_D$ . By the change of variables  $Y = K_D Q_d$ , we obtain

$$\begin{bmatrix} \text{YBLK11} & \text{YBLK12} & \text{YBLK13} & Q_d \\ (\text{YBLK12})^T & \text{QBLK22} & 0 & 0 \\ (\text{YBLK13})^T & 0 & \text{QBLK33} & 0 \\ Q_d^T & 0 & 0 & -H_d \end{bmatrix} < 0 \quad (10)$$

where

$$\text{YBLK11} = \begin{pmatrix} Q_d A_d^T + Y^T B_{u,D}^T + A_d Q_d + B_{u,D} Y \\ + B_{a,D} B_{a,D}^T + B_{p,D} M_d B_{p,D}^T + B_{w,D} N_d B_{w,D}^T \end{pmatrix}$$

$$\text{YBLK12} = (B_{p,D} M_d D_{qp,D}^T + Q_d C_q^T + Y^T D_{qu,D}^T)$$

$$\text{YBLK13} = (B_{w,D} N_d D_{zw,D}^T + Q_d C_{z,D}^T + Y^T D_{zu,D}^T).$$

(10) is LMI in matrix variables  $Q_d$ ,  $Y$ ,  $M_d$  and  $N_d$  and thus can be cast into a convex optimization problem as follows.

$$\begin{cases} \text{Maximize} & \text{Trace } N_d \\ \text{Subject to} & Q_d > 0, N_d > 0, M_d > 0, \text{ and (10).} \end{cases} \quad (11)$$

Once the maximized solutions are found for the problem, then a stabilizing state feedback gain is obtained by

$$K_D = YQ_d^{-1} \quad (12)$$

## V. NUMERICAL EXAMPLES

Consider the system in Fig. 2 consisting of three inverted pendulums of point masses  $m_i$ , and length  $l_i$ . The pendulums interact via three springs and three dampers of stiffness  $k_{ij}$  and damping  $b_{ij}$ ;  $i, j = 1, 2, 3$ , and  $i \neq j$ . The distances from attached point of springs and dampers to the platform baseline are  $a_i$ .

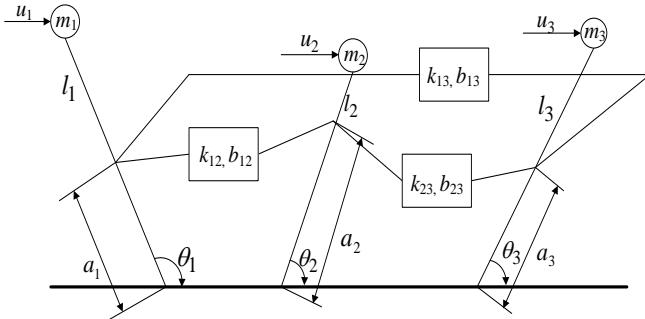


Fig. 2 Three Inverted Pendulums Systems

In this example, we will demonstrate step response for the system using LMI-based decentralized control and show the flexibility of LMI-based approach. In order to achieve zero steady state error, i.e.  $e(t \rightarrow \infty) = 0$ , an integral control will be added as an extra state for each subsystem. The system dynamics for pendulums are written in general form,

$$\ddot{\theta}_i = \frac{1}{m_i} \left\{ \sum_{j=1, j \neq i}^3 \left[ \begin{array}{l} -\left( b_{ij} \dot{\theta}_i + \frac{a_i^2}{l_i^2} k_{ij} \theta_i \right) \\ + \left( \frac{l_j}{l_i} b_{ij} \dot{\theta}_j + \frac{a_i a_j}{l_i^2} k_{ij} \theta_j \right) \\ + \frac{m_i g}{l_i} \theta_i + \frac{1}{l_i} u_i \end{array} \right] \right\} \quad (13)$$

Multiplicative uncertainties are used as shown below,

$$m_i = \underline{m}_i (1 + \Delta_{mi} \delta_{mi}), \quad b_{ij} = \underline{b}_{ij} (1 + \Delta_{bij} \delta_{bij}), \quad k_{ij} = \underline{k}_{ij} (1 + \Delta_{kij} \delta_{kij})$$

in which  $\Delta$  with subscripts, as uncertainty weighting, is the percentage perturbation from nominal value of each parameter and  $\delta$  with subscripts varies from -1 to 1, i.e.  $-1 \leq \delta \leq 1$ . Plant data and weighting of uncertainties are shown in Table 1. Thus, the system can be written as (8). In this example, we let  $D_{zu,D}=0$  to demonstrate a singular case in traditional  $H_\infty$  setup while the control saturation can still be enforced as constraints shown in [7]. A full state-feedback static gain can be obtained by computing feasible solution of (11). If feasible solution does exist, then static gain can be obtained by (12). The simulation was done using Matlab with LMI Control Toolbox and static gain

$$K_D = \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{bmatrix}$$

where 0 represents the zero vector of appropriate dimension,

	$m_1$	$m_2$	$m_3$	$b_{12}$	$b_{13}$	$b_{23}$
nominal	1	1	1	1	.51	1
weighting	.14	.29	.19	1	1	1
	$k_{12}$	$k_{13}$	$k_{23}$	$g=10$		
nominal	1	.51	1	$l_1=1, l_2=1.2, l_3=1.1$		
weighting	1	1	1	$a_1=a_2=a_3=0.5$		

$$K_1 = [-25.0328 \quad -8.0267 \quad -7.2189],$$

$$K_2 = [-28.4028 \quad -10.0355 \quad -7.5556], \text{ and}$$

$$K_3 = [-25.9417 \quad -8.4924 \quad -7.3352].$$

We obtain the overall system poles at

$$-0.9462 \pm j0.2215, \quad -7.6443, \quad -1.0111, \quad -8.6303, \quad -0.7215, \quad -7.3363, \text{ and } -0.9470 \pm j0.1099.$$

Notice that the uncertain data set in Table 1 can be used to form a data convex hull. The extreme points (or vertices) of the convex hull are taken from the corresponding  $\delta$  values, i.e.  $\delta=1$  and  $\delta=-1$ . We assume the system uncertainties vary within this convex hull. It is obvious that the mass weighting in the Table 1 should not be equal to 1 to avoid the possibility that zero mass occurred in the denominator of (7). From the chosen data in the Table 1 we demonstrate the simulation results by varying the uncertainties from one vertex, which the stiffness and damping are twice of their nominal values, to another vertex, which the corresponding stiffness and damping are zero values. This can be done by switching  $\delta$  from 1 to -1. The sudden humps at time 5 sec in Fig. 3 and Fig. 4 show the results due to this change. Notice that it is an instantaneous switch. Next, we turn our attention to examine conservativeness of our design. This can be checked by expanding the vertex of convex hull to see how large it can be to cause instability while keeping the same control gain. To examine every vertex of the convex hull is tedious, thus we will only expand the vertices shown in the simulation as before to get idea of conservativeness. The simulation results show the weighting of stiffness and damping can be simultaneously increased to approximately 3 which shows the design is conservative. In fact, the results are not surprised because Lyapunov function provides only sufficient condition to stabilize a system, which in turn causes a conservative design.

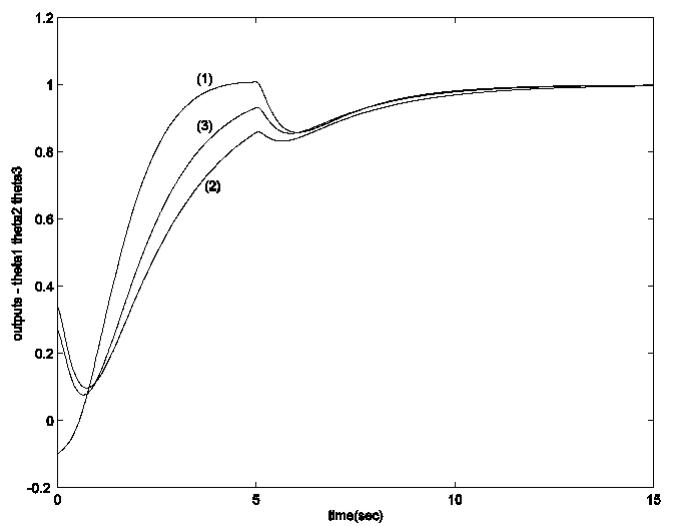


Fig. 3 Outputs of pendulums

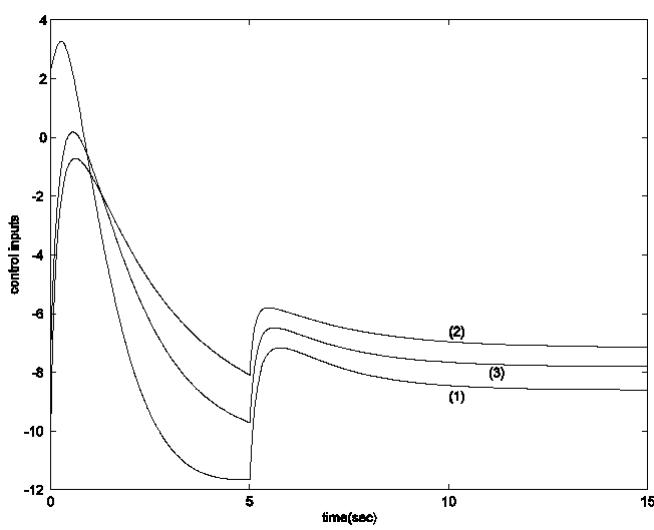


Fig. 4 Control inputs

#### APPENDIX

To prove Theme 1, we will first introduce the following identity and define the notation that will be used in the proof.  
*Identity:*

$$A^T B^T C D + D^T C^T B A = -(D - C^T B A)^T (D - C^T B A) + D^T D + A^T B^T C C^T B A.$$

#### Notations:

The diag( $\bullet$ ) and the usage of subscripts “ $d$ ” and “ $D$ ” have been defined in Section II. The following notations will be used throughout the paper.

$$\Lambda_i = \text{diag}(\lambda_i^r)_{r=1,\dots,nqi}, \quad \Lambda_d = \text{diag}(\Lambda_i)_{i=1,\dots,n}, \\ \Gamma_d = \text{diag}(\gamma_i^2)_{i=1,\dots,n}, \quad C_{z,D} = \text{diag}(C_{z,i}^i)_{i=1,\dots,n},$$

$$\alpha_i = \sum_{k=1, k \neq i}^n 1, \quad \Omega_d = \text{diag}(\alpha_i)_{i=1,\dots,n},$$

$$B_{a,D} = \text{diag}(B_{a,i})_{i=1,\dots,n}, \text{ and}$$

$$B_{a,i} = \begin{bmatrix} A_i^1 & \dots & A_i^{i-1} & 0 & A_i^{i+1} & \dots & A_i^n \end{bmatrix}.$$

#### Proof of Theme 1:

We consider the quadratic Lyapunov function  $V_i(x_i) = x_i^T L_i x_i$ . The consideration of performance and perturbation will be incorporated in the derivative of Lyapunov function for stabilization of overall system. We have

$$\sum_{i=1}^n \left( V_i(x_i) + z_i^T z_i - \gamma_i^2 w_i^T w_i + \sum_{r=1}^{n_{qi}} \lambda_i^r (q_i^r)^T q_i^r - p_i^r p_i^r \right) < 0 \quad (14)$$

Substituting (2) into (14), applying the identity, and rearranging the expression, we obtain the inequality as follows,

$$\sum_{i=1}^n \left( x_i^T \Xi_1^i x_i + \left[ \begin{array}{c} \underline{x} \\ p_i \end{array} \right]^T \Xi_2^i \left[ \begin{array}{c} \underline{x} \\ p_i \end{array} \right] + \left[ \begin{array}{c} x_i \\ p_i \\ w_i \end{array} \right]^T \Xi_3^i \left[ \begin{array}{c} x_i \\ p_i \\ w_i \end{array} \right] - \sum_{j=1, j \neq i}^n (x_j - A_i^{jT} L_i x_i)^T (x_j - A_i^{jT} L_i x_i) \right) < 0, \quad (15)$$

where

$$\Xi_1^i = A_i^{iT} L_i + L_i A_i^i + L_i \left( \sum_{j=1, j \neq i}^n A_i^j A_i^{jT} \right) L_i + \alpha_i I + C_{z,i}^{iT} C_{z,i}^i,$$

$$\Xi_2^i = \begin{bmatrix} C_{q,i}^T \Lambda_i C_{q,i} & C_{q,i}^T \Lambda_i D_{qp,i} \\ D_{qp,i}^T \Lambda_i C_{q,i} & D_{qp,i}^T \Lambda_i D_{qp,i} - \Lambda_i \end{bmatrix}, \text{ and}$$

$$\Xi_3^i = \begin{bmatrix} 0 & L_i B_{p,i} & \Theta_1^i \\ B_{p,i}^T L_i & 0 & 0 \\ (\Theta_1^i)^T & 0 & \Theta_2^i \end{bmatrix} \text{ for } \Theta_1^i = L_i B_{w,i} + C_{z,i}^{iT} D_{zw,i}$$

$$\text{and } \Theta_2^i = D_{zw,i}^T D_{zw,i} - \gamma_i^2 I.$$

By summing up (15), we obtain the inequality

$$\left( \begin{bmatrix} \underline{x} \\ p \\ w \end{bmatrix}_T \begin{bmatrix} \text{BLK11} & \text{BLK12} & \text{BLK13} \\ (\text{BLK12})^T & \text{BLK22} & 0 \\ (\text{BLK13})^T & 0 & \text{BLK33} \end{bmatrix} \begin{bmatrix} \underline{x} \\ p \\ w \end{bmatrix} - \sum_{i=1}^n \sum_{j=1, j \neq i}^n (x_j - A_i^{jT} L_i x_i)^T (x_j - A_i^{jT} L_i x_i) \right) < 0.$$

A sufficient condition for the above inequality to hold requires Theme 1. This completes the proof.

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